

Option Pricing of Twin Assets

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Abstract

How to price and hedge claims on nontraded assets are becoming increasingly important matters in option pricing theory today. The most common practice to deal with these issues is to use another similar or ‘closely related’ asset or index which is traded, for hedging purposes. Implicitly, traders assume here that the higher the correlation between the traded and non-traded assets, the better the hedge is expected to perform. This raises the question as to how ‘closely related’ the assets really are. In this paper, the concept of twin assets is introduced, focusing the discussion precisely in what does it mean for two assets to be similar. Our findings point to the fact that, in order to have very similar assets, for example identical twins, high correlation measures are not enough. Specifically, two basic criteria of similarity are pointed out: i) the coefficient of variation of the assets and ii) the correlation between assets. From here, a method to measure the level of similarity between assets is proposed, and secondly, an option pricing model of twin assets is developed. The proposed model allows us to price an option of one nontraded asset using its twin asset, but this time knowing explicitly what levels of errors we are facing. Finally, some numerical illustrations show how twin assets behave depending upon their levels of similarities, and how their potential differences will traduce in MAPE (mean absolute percentage error) for the proposed option pricing model.

Keywords: Cross Hedging, Twin Assets, Option Pricing.

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1 Introduction

Usually, financial markets in the real world are not complete. Indeed, for instance, how to price and hedge claims on nontraded assets are becoming increasingly important matters in option pricing theory today. The most common practice to deal with this issue is to use for hedging purposes another similar or ‘closely related’ asset or index which is traded. The finance literature has focused on finding the best hedging strategy, usually using a utility maximization approach, see for example [6, 5]. It is well known that this cross hedging mechanism in an incomplete financial market creates what has been called basis risk, see [3, 1].

In this line of research, implicitly traders assume here that the higher the correlation between the traded and nontraded assets, the better the hedge is expected to perform. This is certainly an advantage of this strategy today, since over the past ten years, cross-asset correlations roughly doubled in international financial markets¹. On the other hand, there are many examples of financial problems of this kind, with one nontraded assets, where cross hedging is needed: for example an exporter of a particular commodity when the hedging instrument relates to another commodity, [8], or commodity-currency cross-hedges, see [4].

Nevertheless, cross hedging in an incomplete financial market raises the question as to what really are similar or ‘closely related’ assets, beyond the high correlation between the traded and nontraded assets. In fact, there are not many studies analysing the effectiveness of this definition of assets similarity. In this paper, we introduce the concept of twin assets, focusing the discussion precisely in what does it mean to be similar. Our findings point to the fact that, in order to have very similar assets, for example identical twins following our metaphor, high correlation measures are not enough, we need to consider in our characterization of similarity, a normalise measure of relative volatility. Specifically, we consider two basic criteria of similarity to define twin assets: i) the coefficient of variation of the assets, and ii) the correlation between assets.

The structure of the paper is as follows. In section 2, firstly, a method to measure the level of similarity between assets is proposed, and secondly, an option pricing model of twin assets is developed. The proposed model allow us to price an option of one nontraded asset using its twin asset, but this time knowing explicitly what levels of errors we are facing. In section 3, some numerical illustrations show how twin assets behave depending upon their levels of similarities, and how their potential differences will traduce in MAPE (mean absolute percentage error) for the proposed option pricing model. Finally, some conclusions and future research are presented.

¹JPMorgan developed an analysis on the correlation between 45 developed world and emerging market country equity benchmarks contained in the MSCI All Country World Index, see [7]. Its results show that, over the past 20 years, the average correlation between these country benchmarks roughly doubled, from 30% to 60%.

2 The Model

In this section our model is presented. Firstly, we introduce two parameters than help us to measure the level of similarity between assets. Latter, with the definition of twin assets, an option pricing model is developed, considering implicitly the parameters of similarities mentioned above.

2.1 Looking for Twin Assets

Let S_i^t be the value of the stock i and S_j^t the value of the stock j , both at time t . Let us assume that they are governed by the following stochastic differential equations:

$$dS_i^t = \mu_i S_i^t dt + \sigma_i S_i^t dW_i^t \quad (1)$$

$$dS_j^t = \mu_j S_j^t dt + \sigma_j S_j^t dW_j^t \quad (2)$$

where σ_i and σ_j are constants. W_i^t and W_j^t are Gauss-Wiener processes that maintain the following relationship:

$$dW_i^t \cdot dW_j^t = \rho dt \quad (3)$$

The last equation implies that returns of stock i and j are correlated, and that the value of that correlation is ρ . An equivalent expression for (3) is:

$$dW_i^t = \rho dW_j^t + \sqrt{1 - \rho^2} d\tilde{W}^t \quad (4)$$

with \tilde{W}^t defined as an independent Gauss-Wiener processes.

Until here, we have followed the standard procedure for modelling similar assets. Nevertheless, when two highly correlated assets are modelled, they are not necessarily going to follow exactly the same trajectory, neither, they are going to finish in a similar point. The only thing we can say about them is that they are going to go up and down at the same time, but the amount of the ups and downs will depend upon the volatility. On the other hand, the volatility has to be corrected by the mean (two assets with equal volatilities but different means are not going to follow similar trajectories neither). So we need to include both the volatility and the mean as variables in order to define similar assets. Thus, we introduce a normalise measure of relative volatility, an α parameter, defined by quotient of the the coefficients of variation of each asset, which is given by the following expression:

$$\alpha \equiv \frac{(c_v)_i}{(c_v)_j} = \frac{\sigma_i \mu_j}{\sigma_j \mu_i} \quad (5)$$

Then, using the Itô's $\frac{1}{2}$'s calculus, we obtain an analytical solution for S_i and S_j at time $T > t'$:

$$S_i^T = S_i^{t'} e^{(\mu_i - \frac{1}{2}\sigma_i^2)(T-t') + \sigma_i(\rho W_j^{T-t'} + \sqrt{1-\rho^2}\tilde{W}^{T-t'})} \quad (6)$$

$$S_j^T = S_j^{t'} e^{(\mu_j - \frac{1}{2}\sigma_j^2)(T-t') + \sigma_j W_j^{T-t'}} \quad (7)$$

where $S_i^{t'}$ and $S_j^{t'}$ are given values at the time t' of the stock i and j respectively.

From (6) and (7) we can build expressions for the returns of the stocks S_i and S_j , called R_i and R_j respectively, over the time span $T - t'$:

$$R_i^{T-t'} = \ln \left(\frac{S_i^T}{S_i^{t'}} \right) \quad (8)$$

$$R_j^{T-t'} = \ln \left(\frac{S_j^T}{S_j^{t'}} \right) \quad (9)$$

After replacing and rearranging some terms we have:

$$R_j^{T-t'} = \alpha \frac{\sigma_j}{\sigma_i} R_i^{T-t'} + \frac{1}{2} \sigma_j (\alpha \sigma_i - \sigma_j) (T - t') + \sigma_j (1 - \rho \alpha) W_j^{T-t'} - \alpha \sigma_j \sqrt{1 - \rho^2} \tilde{W}^{T-t'} \quad (10)$$

and hence:

$$S_j^T e^{\sigma_j(1-\rho\alpha)W_j^{T-t'} - \alpha\sigma_j\sqrt{1-\rho^2}\tilde{W}^{T-t'}} = S_j^{t'} \left(\frac{S_i^T}{S_i^{t'}} \right)^{\alpha \frac{\sigma_j}{\sigma_i}} e^{\frac{1}{2}\sigma_j(\alpha\sigma_i - \sigma_j)(T-t')} \quad (11)$$

Equations (10) and (11) give relationships between each pair of returns and stocks respectively, allowing us to express the stock (return) j in terms of the stock (return) i . These equalities, given their stochastic nature, can be modelled only when the stochastic terms (i.e. $W_j^{T-t'}$ and $\tilde{W}^{T-t'}$) are known. Therefore, we propose the following approximation:

$$S_j^T \approx S_j^{t'} \left(\frac{S_i^T}{S_i^{t'}} \right)^{\alpha \frac{\sigma_j}{\sigma_i}} e^{\frac{1}{2}\sigma_j(\alpha\sigma_i - \sigma_j)(T-t')} e^{\sigma_j(1-\rho\alpha)W_x^{T-t'} - \alpha\sigma_j\sqrt{1-\rho^2}W_y^{T-t'}} \quad (12)$$

where $W_x^{T-t'}$ and $W_y^{T-t'}$ are independent Brownian motions.

We see that (12) is governed by the values of ρ and α . If simultaneously $\rho \rightarrow 1$ and $\alpha \rightarrow 1$ the error of the approximation will be the minimum. Thus, we will define as *twin assets*, when both ρ and α parameters come close to the unity (i.e. the returns of the assets are totally correlated and have the same (similar) coefficients of variation). Alternatively, we can have different levels of similarity depending upon the values of ρ and α .⁴

Finally, the expression (12) can be written in a more compact form as:

$$S_j^T \approx AB (S_i^T)^{\alpha \frac{\sigma_j}{\sigma_i}} \quad (13)$$

being A a deterministic term and B a stochastic term given by:

$$A = S_j^{t'} \left(S_i^{t'} \right)^{-\alpha \frac{\sigma_j}{\sigma_i}} e^{\frac{1}{2} \sigma_j (\alpha \sigma_i - \sigma_j) (T - t')} \quad (14)$$

$$B = e^{\sigma_j (1 - \rho \alpha) W_x^{T - t'} - \alpha \sigma_j \sqrt{1 - \rho^2} W_y^{T - t'}} \quad (15)$$

From these equations we can clearly see that the correlation between assets is a necessary but not sufficient condition in order to have similar or twin assets. As we will see in the next sections, the predictive performance of option pricing and hedging will depend crucially on the ρ and α parameters.

2.2 On the Option Pricing of Twin Assets

An important practical question for traders and investors in the presence of incomplete markets is how to price and hedge what we have called twin options. For example, in the case of a nontraded asset or a given option, if we have a twin asset, defined by their similarities in terms of the coefficient of variation and correlation, we could have a potential pricing and hedging of the asset or option, but this time with a measure of the errors involved.

According with the traditional approach, a derivative c_j with underlying asset S_j follow the Black-Scholes equation [2]:

$$rc_j = \frac{\partial c_j}{\partial t} + \frac{1}{2} \sigma_j^2 S_j^2 \frac{\partial^2 c_j}{\partial S_j^2} + r S_j \frac{\partial c_j}{\partial S_j} \quad (16)$$

Being r the risk-free rate of interest for a term.

If we consider c_j as a vanilla call option with time to maturity T (fixed the starting date of the contract in $t = 0$) and Strike Price K_j , the value of c_j is given by the Black-Scholes formula:

$$c_j(T - t) = S_j^t N(d_{1_j}) - K_j e^{-r(T - t)} N(d_{2_j}) \quad (17)$$

where $N(\cdot)$ is the cumulative normal density function and

$$d_{1_j} = \frac{\ln \left(\frac{S_j^t}{K_j} \right) + \left(r + \frac{\sigma_j^2}{2} \right) (T - t)}{\sigma_j \sqrt{T - t}} \quad (18)$$

$$d_{2_j} = \frac{\ln \left(\frac{S_j^t}{K_j} \right) + \left(r - \frac{\sigma_j^2}{2} \right) (T - t)}{\sigma_j \sqrt{T - t}} \quad (19)$$

On the another hand, we can link c_j with an option over the stock S_i . Since $A > 0$ and $B > 0$, by (13) and considering the payoff function of c_j :

$$\{S_j^T - K_j\}^+ \approx AB \left\{ (S_i^T)^{\alpha \frac{\sigma_j}{\sigma_i}} - K_i \right\}^+ \quad (20)$$

where $K_i = K_j / (AB)$.

Now, using (20) the price of c_j also is related with the value of S_i . By risk neutral valuation, we have:

$$e^{-r(T-t)} \mathbb{E} \left[\{S_j^T - K_j\}^+ \right] \approx e^{-r(T-t)} \mathbb{E} \left[AB \left\{ (S_i^T)^{\alpha \frac{\sigma_j}{\sigma_i}} - K_i \right\}^+ \right] \quad (21)$$

The left side of (21) correspond to the price of c_j at time $T - t$ (see (17)). Then,

$$c_j \approx e^{-r(T-t)} \int_{K_i^{\frac{\sigma_j}{\alpha \sigma_j}}}^{\infty} \left((S_i^T)^{\alpha \frac{\sigma_j}{\sigma_i}} - K_i \right) l(S_i^T) dS_j^T \quad (22)$$

being $l(S_i^T)$ the probability distribution of S_i^T . Using (6) and the fact that $W_i^T = w\sqrt{(T-t)}$, with $w \sim f(w)$, being f the normal standard distribution²; (22) is transformed into:

$$c_i(T-t) \approx AB \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-g_{2i}}^{\infty} \left[\left(S_i^t e^{(r - \frac{1}{2}\sigma_i^2)(T-t) + x\sigma_i\sqrt{T-t}} \right)^{\alpha \frac{\sigma_j}{\sigma_i}} - K_i \right] e^{-\frac{w^2}{2}} dw \quad (23)$$

where,

$$g_2 = \frac{\ln \left(\frac{S_i^t}{K_i^{\sigma_i/\alpha\sigma_j}} \right) + (r - \frac{1}{2}\sigma_i^2)(T-t)}{\sigma_i\sqrt{T-t}} \quad (24)$$

In order to solve, we separate the left side into two integrals:

$$c_i(T-t) \approx I_1 - I_2 \quad (25)$$

being,

²i.e. w is standard normal variable

$$I_1 = \frac{AB(S_i^t)^{\alpha \frac{\sigma_j}{\sigma_i}} e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-g_{2_i}}^{\infty} e^{[(r - \frac{1}{2}\sigma_i^2)(T-t) + x\sigma_i\sqrt{T-t}] \alpha \frac{\sigma_j}{\sigma_i}} e^{-\frac{w^2}{2}} dw \quad (26)$$

and

$$I_2 = \frac{ABK_i e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-g_{2_i}}^{\infty} e^{-\frac{w^2}{2}} dw \quad (27)$$

Now, we only need to solve I_1 and I_2 to find the value of c_j in terms of S_i . After that, we have:

$$c_j \approx AB(S_i^t)^{\alpha \frac{\sigma_j}{\sigma_i}} e^{(\alpha \frac{\sigma_j}{\sigma_i} - 1)(r + \frac{1}{2}\alpha\sigma_j\sigma_i)(T-t)} N(g_{1_i}) - ABK_i e^{-r(T-t)} N(g_{2_i}) \quad (28)$$

where

$$g_1 = \frac{\ln\left(\frac{S_i^t}{K_i^{\sigma_i/\alpha\sigma_j}}\right) + \left[r + \left(\alpha \frac{\sigma_j}{\sigma_i} - \frac{1}{2}\right)\sigma_i^2\right](T-t)}{\sigma_i\sqrt{T-t}} \quad (29)$$

$$= g_2 + \alpha\sigma_j\sqrt{T-t} \quad (30)$$

Finally, and written in complete form, we have the option valuation of asset j using information of its twin asset i :

$$\begin{aligned} c_j \approx & S_j^t e^{[(\alpha \frac{\sigma_j}{\sigma_i} - 1)(r + \frac{1}{2}\alpha\sigma_j\sigma_i) + \frac{1}{2}\sigma_j(\alpha\sigma_i - \sigma_j)](T-t)} e^{\sigma_j(1-\rho\alpha)W_x^{T-t} - \alpha\sigma_j\sqrt{1-\rho^2}W_y^{T-t}} N(g_1) \\ & - S_j^t (S_i^t)^{-\alpha \frac{\sigma_j}{\sigma_i}} K_i e^{[\frac{1}{2}\sigma_j(\alpha\sigma_i - \sigma_j) - r](T-t)} e^{\sigma_j(1-\rho\alpha)W_x^{T-t} - \alpha\sigma_j\sqrt{1-\rho^2}W_y^{T-t}} N(g_2) \end{aligned} \quad (31)$$

From equation 31 we can see that in order to value an option of the original asset called j , we need to have the following parameters: μ_j , μ_i , σ_j , σ_i , and ρ . Besides we need the initial values of S . The α parameter will be implicit in the μ 's and σ 's. On the other hand, the goodness of fit of the pricing will be given in terms of the parameters ρ and α .

3 Numerical Illustration

In this section, we develop a numerical illustration using the following parameters $\mu_i = 0.4$, $\mu_j = 0.8$, $\sigma_i = 0.2$, $S_i^{t=0} = 80$ and $S_j^{t=0} = 90$. Firstly, given these two assets we will model one asset price path using its twin with different values of ρ and α , indicating its predictive power through its mean absolute percentage error (MAPE). Secondly, using the same example we will use our new option pricing formula and we will evaluate its performance to predict a 3 month call option, again for different values of ρ and α .

3.1 Twin Assets

We simulate, under several values of ρ and α^3 , the path of the two stocks involved, S_i and S_j (Eqs. 6 and 7); where S_i is the twin of S_j ; and compute, using (13), the model prediction for one day at future. These results are plotting in Fig. 1.

We see in the Fig. 1, a perfect fit when $(\rho, \alpha) = (1, 1)$. We also see that if we have a low value of correlation, the fit is least precise. On the other hand, if $\alpha \neq 1$ the results lose accuracy. In order to have a more clear and systematic measure of error, we use a Montecarlo simulation, with 40000 replications for each pair of values (ρ, α) , and then we compute the Mean Absolute Percentage Error (MAPE) of these simulations. The MAPE was obtained using the following definition:

$$\text{MAPE}(\rho_l, \alpha_m) = \frac{100}{N} \sum_{n=1}^N \left| \frac{S'_{jn}(\rho_l, \alpha_m) - S_{jn}}{S_{jn}} \right| \quad (32)$$

where $S'_{jn}(\rho_l, \alpha_m)$ is the value of the n^{th} simulation of the asset S_j using it twin (given by 12) with parameters ρ_l and α_m ; S_{jn} is the n^{th} simulation of the asset S_j , and N is the number of replications. The results of these simulations are presented in 2. At the right side of the figure a fix value of α is shown; in this case the MAPE is monotonically decreasing function (in relation top). In case of $(\rho, \alpha) = (1, 1)$, we have a perfect twin (without error).

If we would estimate the error for another time at future, the MAPE have the same form as above, but the values are obviously amplified (see 3).

Besides, the error (MAPE) is clearly related with the value of σ_j . If the value of σ_j increases, the error will grow. We can see that behavior in fig.4. At the left side of the image we have a $\rho = 1$ and different values of α ; and at the right side we fix $\alpha = 1$ and different values of ρ . We plot the MAPE for three different values of σ_j . In the two subplots, the MAPE increases if σ_j is greater.

3.2 Option Pricing of Twin Assets

Similar to the previous section, we shown in Fig. 5 the MAPE of the proposed option pricing model. For this example we use the same parameters of the above subsection, besides $K_J = S_j^{t=0}$ (exercise price) and $r = 0.05$ (risk free rate). In the same way, these errors were obtained using the same methodology than before:

$$\text{MAPE2}(\rho_l, \alpha_m) = \frac{100}{N} \sum_{n=1}^N \left| \frac{c'_{jn}(\rho_l, \alpha_m) - c_j}{c_j} \right| \quad (33)$$

³The value of α was modified by changing μ_j

being c_j the theoretical value of a call option over the stock S_j with strike price K_J (see 17), $c'_{j_n}(\rho_l, \alpha_m)$ the n^{th} simulation of c_j using the relation (31) with parameters ρ_l and α_m ; S_{j_n} is the n^{th} simulation of the asset S_j , and $N = 10000$.

As might be expected, the minimum error occurred when both ρ and α are close to one. In fact, only when $\rho = 1$ and $\alpha \in [0.95, 1.05]$ the MAPE was under 10%. A MAPE under 40% occurred when $\rho = 1$ and $\alpha \in [0.8, 1.25]$. In general for values of (ρ, α) distant to $(1, 1)$, the error value is very high, over 100% for instance. As shown above, the MAPE depends on the level of sigma, being proportional in this case to the value of σ_j . In this particular case, negative and positive ρ are not symmetric in terms of error, when α is kept constant, in fact the error is decreasing when ρ increases.

4 Conclusions

How to price and hedge claims on nontraded assets are becoming increasingly important matters. Indeed, cross hedging strategies are widely used in option pricing practice today. Nevertheless, there are not many analytical studies about the goodness of fit of these methods.

In this paper, the concept of twin assets was introduced, focusing the discussion precisely in what does it mean for two assets to be similar. Our findings point to the fact that, in order to have very similar assets, for example identical twins assets, high correlation measures are not enough. Specifically, two basic criteria of similarity are pointed out: i) the coefficient of variation of the assets and ii) the correlation between assets, these parameters may be considered as a measure of accuracy and precision respectively.

An option pricing model of twin assets is also developed, allowing us to price an option of one nontraded asset using information of its twin asset, but this time knowing explicitly what theoretical levels of errors we are facing. Our numerical illustrations show, as might be expected, that the minimum error in option pricing occurred when both correlation (ρ) and the ratio of the coefficient of variation (α) of the assets are close to one.

The empirical calibration of the model for real situations is an interested future research. Also a more realistic, but more complex analysis, could consider the stochastic modelling of parameters ρ and α .

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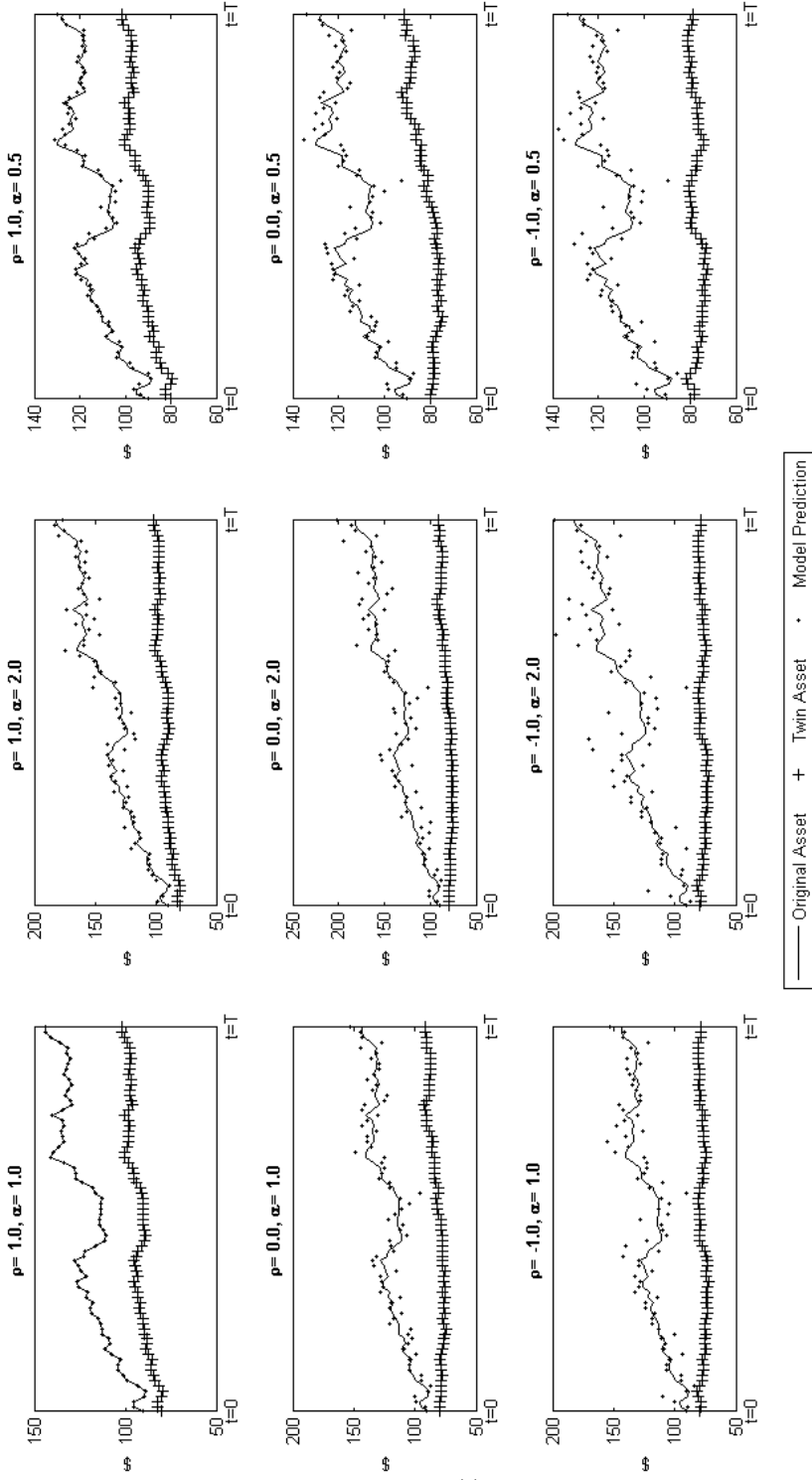


Figure 1: Model prediction, for one asset using it twin with different values of ρ and α . We using $\mu_i = 0.4$, $\mu_j = 0.8$, $\sigma_i = 0.2$, $S_i^{t=0} = 80$ and $S_j^{t=0} = 90$

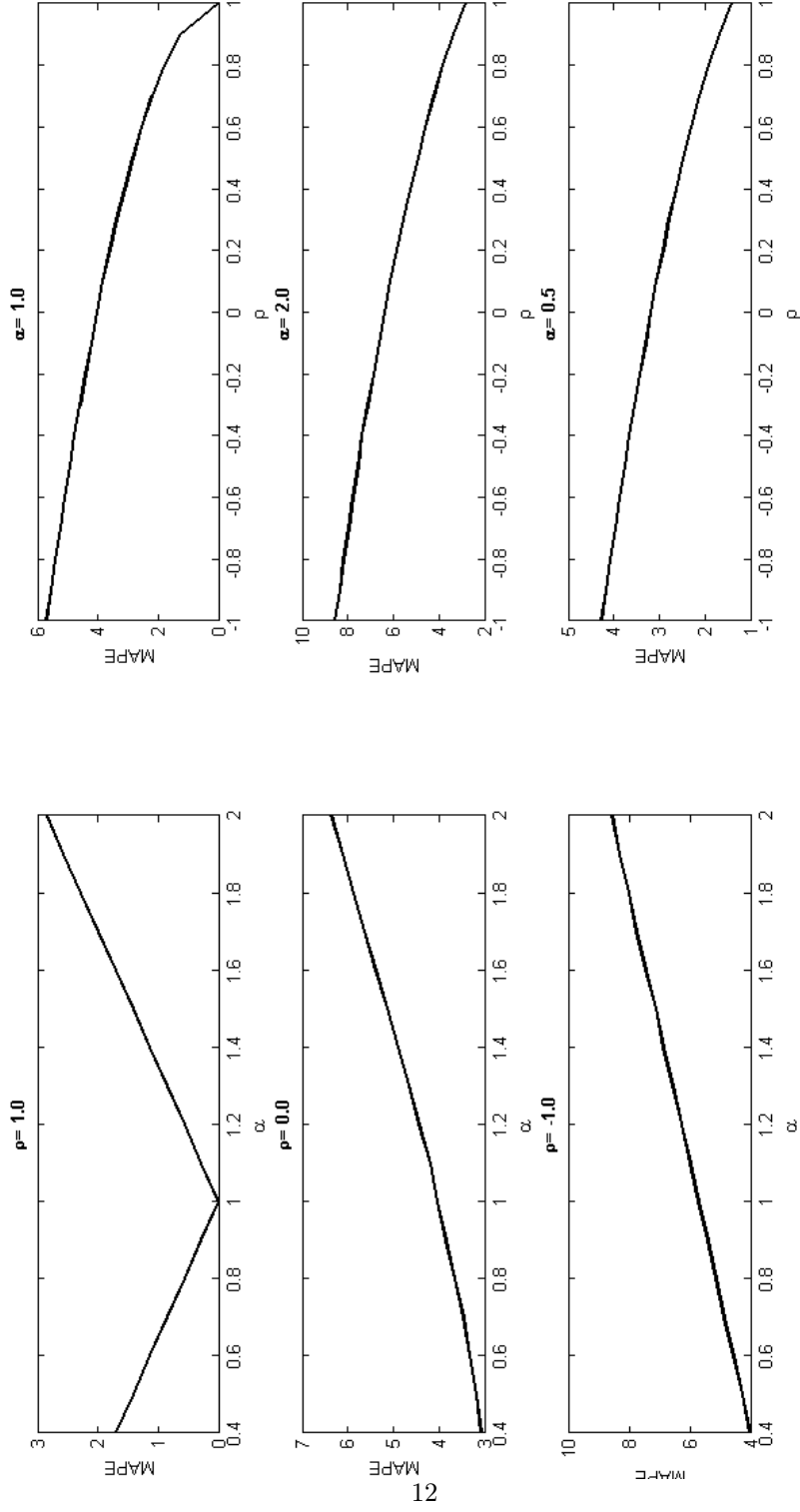


Figure 2: Mean absolute percentage error for the Monte Carlo model prediction, for one day, of one asset using it twin.

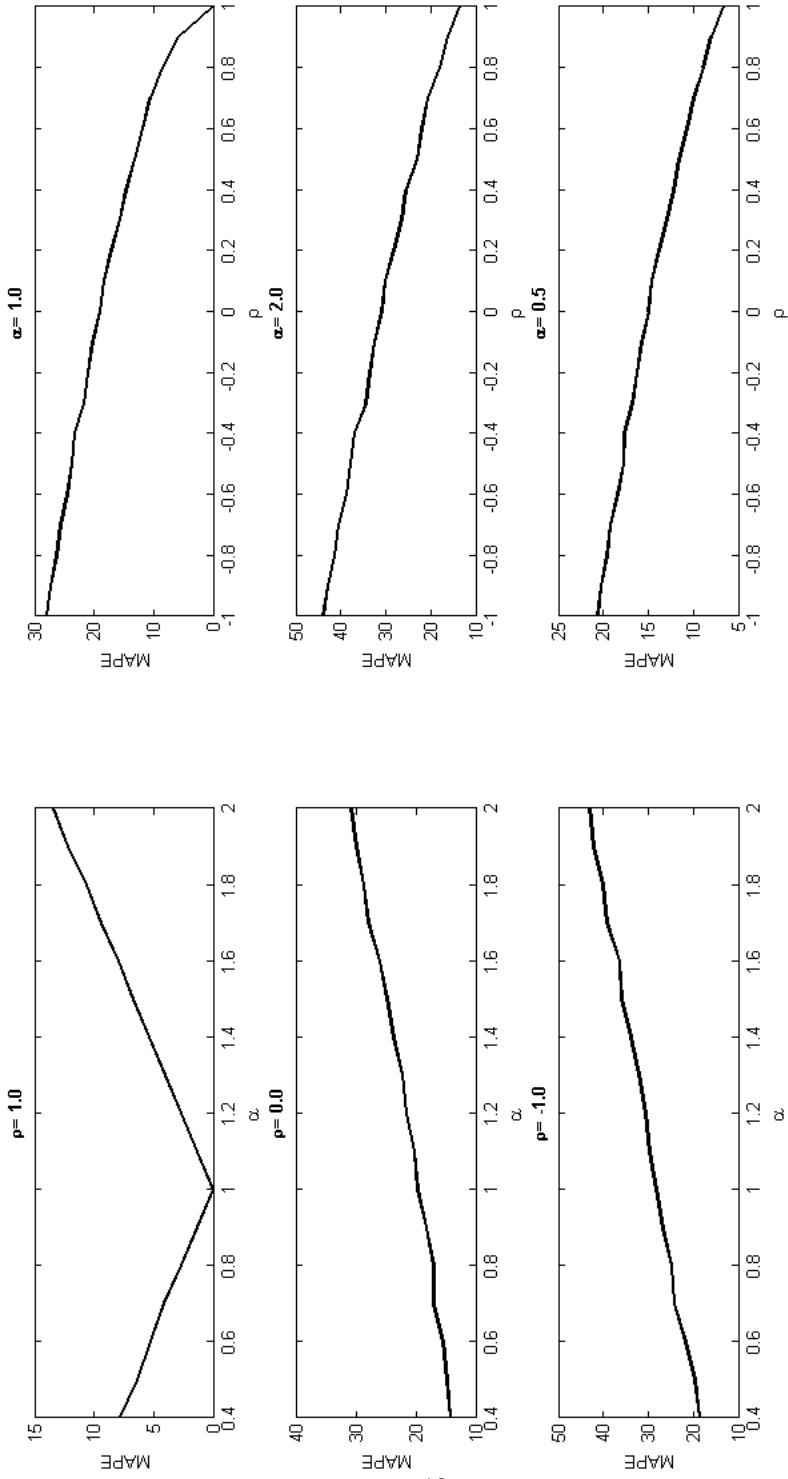


Figure 3: Mean absolute percentage error for the Montecarlo model prediction, for one month, of one asset using it twin.

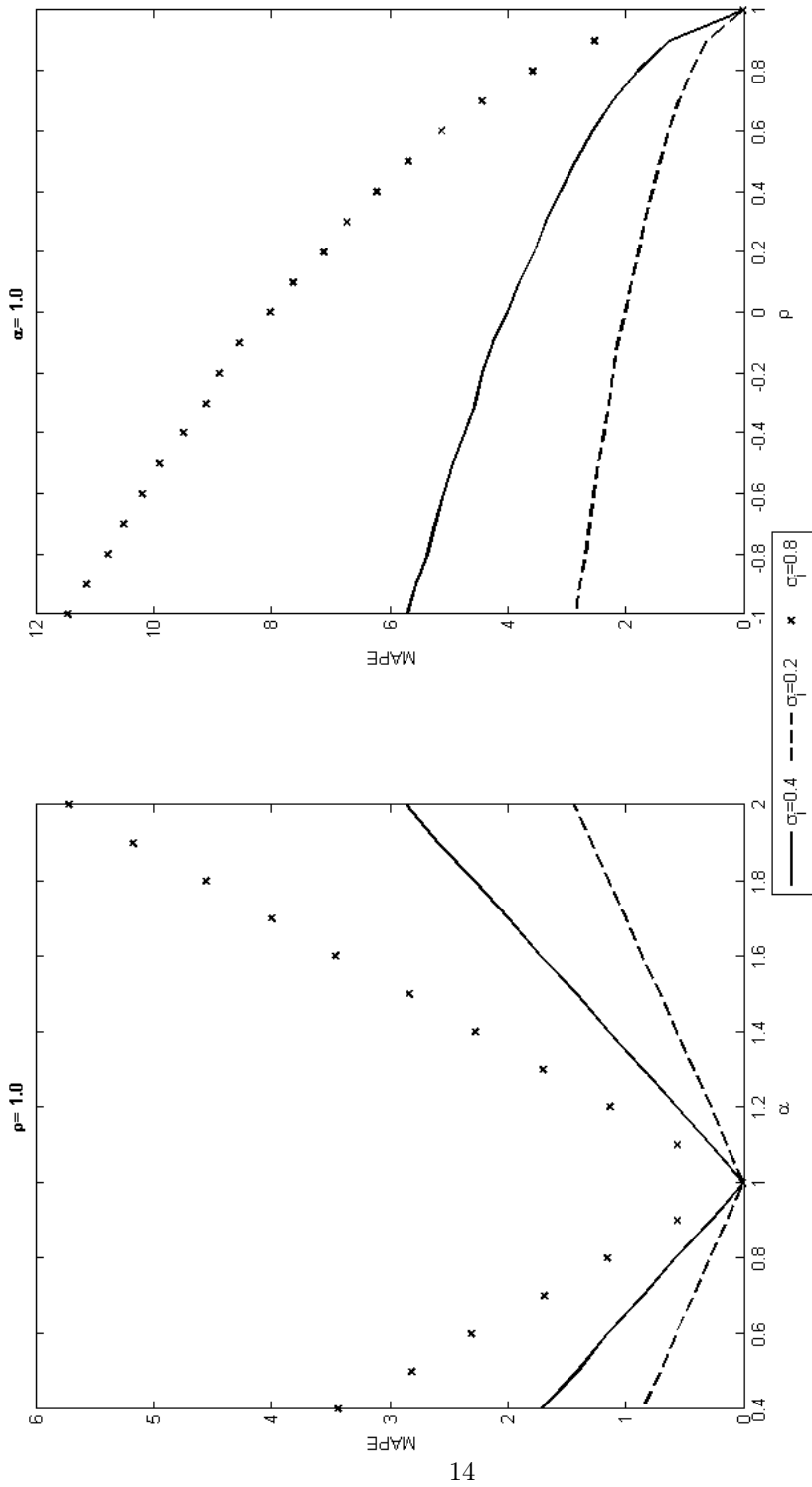


Figure 4: Mean absolute percentage error for the Montecarlo model prediction, for one day, of one asset using it twin, using different values of σ_j .

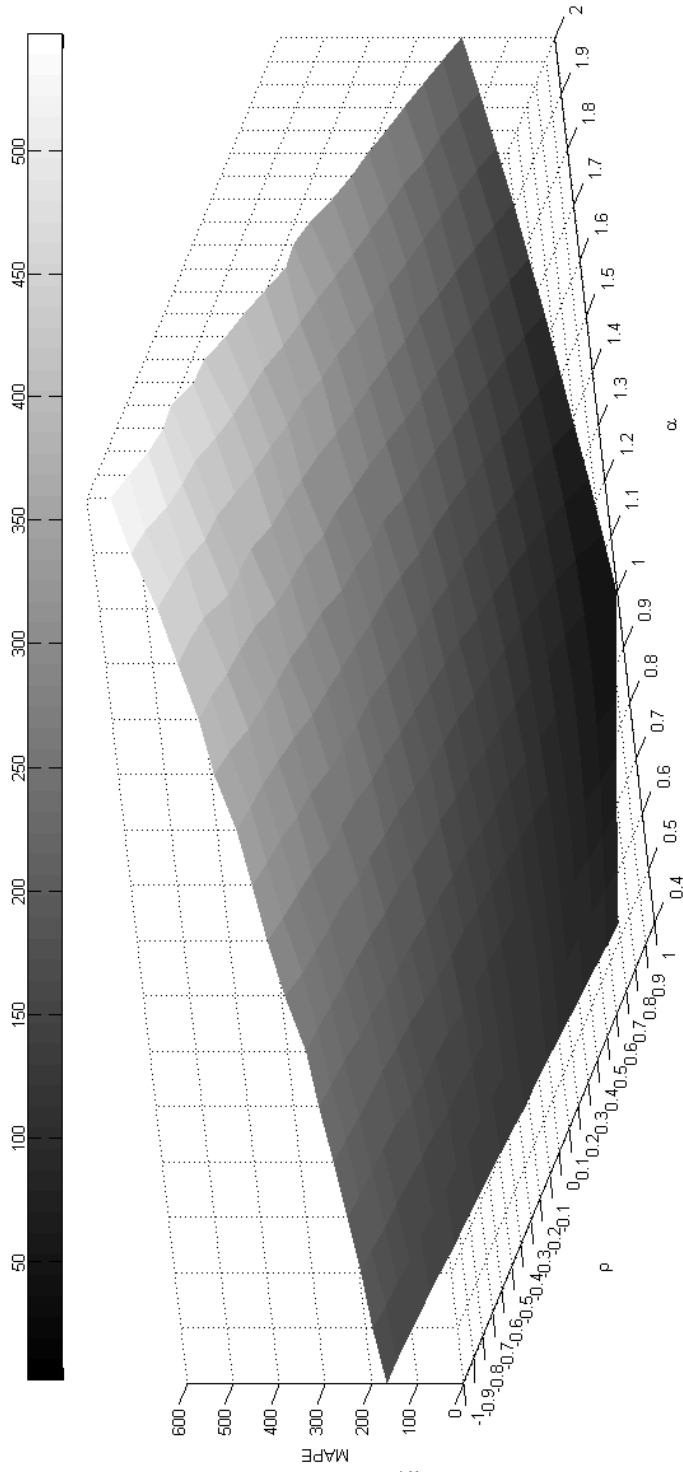


Figure 5: Mean absolute percentage error for the Montecarlo model prediction, for a call option with three months of maturity, using the twin asset approach.